Exercise 3.4.13

Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to

$$u(0,t) = A(t), \quad u(L,t) = 0, \quad \text{and} \quad u(x,0) = g(x).$$

Assume that $u(x,t)$ has a Fourier sine series. Determine a differential equation for the Fourier coefficients (assume appropriate continuity).

Solution

Start by making the substitution,

$$w(x,t) = u(x,t) - A(t)\left(1 - \frac{x}{L}\right),$$

so that both boundary conditions become homogeneous.

$$\begin{array}{lclcl} w(0,t) = u(0,t) - A(t)(1) & \rightarrow & w(0,t) = A(t) - A(t) & \rightarrow & w(0,t) = 0 \\ w(L,t) = u(L,t) - A(t)(0) & \rightarrow & w(L,t) = 0 - 0 & \rightarrow & w(L,t) = 0 \end{array}$$

Solve the substitution for u.

$$u(x,t) = w(x,t) + A(t)\left(1 - \frac{x}{L}\right)$$

Determine $\partial u/\partial t$ and $\partial^2 u/\partial x^2$.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial w}{\partial t} + A'(t)\left(1 - \frac{x}{L}\right) \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 w}{\partial x^2} \end{aligned}$$

Substitute these formulas into the PDE.

$$\frac{\partial w}{\partial t} + A'(t)\left(1 - \frac{x}{L}\right) = k\frac{\partial^2 w}{\partial x^2}$$

The PDE that w satisfies is then

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} + A'(t)\left(\frac{x}{L} - 1\right).$$

In order for the homogeneous Dirichlet boundary conditions to be satisfied, we assume the solution has the form of a Fourier sine series.

$$w(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{L} \tag{1}$$

Since w is continuous, this is justified. Apply the initial condition now to determine $B_n(0)$.

$$\begin{aligned} w(x,0) &= u(x,0) - A(0)\left(1 - \frac{x}{L}\right) \\ \sum_{n=1}^{\infty} B_n(0)\sin\frac{n\pi x}{L} &= g(x) - A(0)\left(1 - \frac{x}{L}\right) \end{aligned}$$

Multiply both sides by $\sin \frac{p\pi x}{L}$, where p is an integer.

$$\sum_{n=1}^{\infty} B_n(0) \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L} = \left[g(x) - A(0)\left(1 - \frac{x}{L}\right)\right] \sin \frac{p\pi x}{L}$$

Integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} B_n(0) \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx = \int_0^L \left[g(x) - A(0)\left(1 - \frac{x}{L}\right)\right] \sin \frac{p\pi x}{L}\, dx$$

Split up the integral on the left and bring the constants in front.

$$\sum_{n=1}^{\infty} B_n(0) \int_0^L \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx = \int_0^L \left[g(x) - A(0)\left(1 - \frac{x}{L}\right)\right] \sin \frac{p\pi x}{L}\, dx$$

The sine functions are orthogonal, so the integral on the left is zero if $n \neq p$. Only if $n = p$ is it nonzero.

$$B_n(0) \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = \int_0^L \left[g(x) - A(0)\left(1 - \frac{x}{L}\right)\right] \sin \frac{n\pi x}{L}\, dx$$

Evaluate the integral on the left.

$$B_n(0)\left(\frac{L}{2}\right) = \int_0^L \left[g(x) - A(0)\left(1 - \frac{x}{L}\right)\right] \sin \frac{n\pi x}{L}\, dx$$

Solve for $B_n(0)$.

$$\begin{aligned}
B_n(0) &= \frac{2}{L} \int_0^L \left[g(x) - A(0)\left(1 - \frac{x}{L}\right)\right] \sin \frac{n\pi x}{L}\, dx \\
&= \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx - \frac{2}{L} A(0) \int_0^L \left(1 - \frac{x}{L}\right) \sin \frac{n\pi x}{L}\, dx \\
&= \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx - \frac{2}{n\pi} A(0)
\end{aligned}$$

This formula for $B_n(0)$ will be needed later. Assuming $\partial w/\partial t$ is continuous, term-by-term differentiation with respect to t in equation (1) is valid.

$$\frac{\partial w}{\partial t} = \sum_{n=1}^{\infty} B_n'(t) \sin \frac{n\pi x}{L}$$

Because w is continuous and $w(0,t) = w(L,t) = 0$, differentiation of the sine series with respect to x in equation (1) is valid as well.

$$\frac{\partial w}{\partial x} = \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos \frac{n\pi x}{L}$$

$\partial u/\partial x$ is continuous, so differentiation of this cosine series with respect to x is justified.

$$\frac{\partial^2 w}{\partial x^2} = \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{L^2}\right) B_n(t) \sin \frac{n\pi x}{L}$$

Substitute these infinite series into the PDE for w.

$$\sum_{n=1}^{\infty} B_n'(t)\sin\frac{n\pi x}{L} = k\sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{L^2}\right)B_n(t)\sin\frac{n\pi x}{L} + A'(t)\left(\frac{x}{L}-1\right)$$

Bring them both to the left side.

$$\sum_{n=1}^{\infty} B_n'(t)\sin\frac{n\pi x}{L} + k\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2}{L^2}\right)B_n(t)\sin\frac{n\pi x}{L} = A'(t)\left(\frac{x}{L}-1\right)$$

Combine them and factor the summand.

$$\sum_{n=1}^{\infty}\left[B_n'(t) + \frac{kn^2\pi^2}{L^2}B_n(t)\right]\sin\frac{n\pi x}{L} = A'(t)\left(\frac{x}{L}-1\right)$$

To obtain the quantity in square brackets, multiply both sides by $\sin\frac{p\pi x}{L}$, where p is an integer,

$$\sum_{n=1}^{\infty}\left[B_n'(t) + \frac{kn^2\pi^2}{L^2}B_n(t)\right]\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L} = A'(t)\left(\frac{x}{L}-1\right)\sin\frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L\sum_{n=1}^{\infty}\left[B_n'(t) + \frac{kn^2\pi^2}{L^2}B_n(t)\right]\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx = \int_0^L A'(t)\left(\frac{x}{L}-1\right)\sin\frac{p\pi x}{L}\,dx$$

Split up the integral on the left and bring the constants in front on both sides.

$$\sum_{n=1}^{\infty}\left[B_n'(t) + \frac{kn^2\pi^2}{L^2}B_n(t)\right]\int_0^L\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx = A'(t)\int_0^L\left(\frac{x}{L}-1\right)\sin\frac{p\pi x}{L}\,dx$$

Since the sine functions are orthogonal, the integral on the left is zero if $n \neq p$. Only if $n = p$ is it nonzero.

$$\left[B_n'(t) + \frac{kn^2\pi^2}{L^2}B_n(t)\right]\int_0^L\sin^2\frac{n\pi x}{L}\,dx = A'(t)\int_0^L\left(\frac{x}{L}-1\right)\sin\frac{n\pi x}{L}\,dx$$

Evaluate the integrals.

$$\left[B_n'(t) + \frac{kn^2\pi^2}{L^2}B_n(t)\right]\frac{L}{2} = A'(t)\left(-\frac{L}{n\pi}\right)$$

Multiply both sides by $2/L$.

$$B_n'(t) + \frac{kn^2\pi^2}{L^2}B_n(t) = -\frac{2}{n\pi}A'(t)$$

This is a first-order linear ODE, so it can be solved by using an integrating factor I.

$$I = \exp\left(\int^t\frac{kn^2\pi^2}{L^2}\,ds\right) = \exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

Multiply both sides of the ODE by I.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n'(t) + \frac{kn^2\pi^2}{L^2}\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t) = -\frac{2}{n\pi}A'(t)\exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

The left side can be written as $d/dt(IB_n)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t)\right] = -\frac{2}{n\pi}A'(t)\exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

Integrate both sides with respect to t.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t) = -\frac{2}{n\pi}\int^t A'(s)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C$$

The lower limit of integration is arbitrary, so it will be set to zero. C will be adjusted to account for any choice that's made. Use integration by parts on the right side to simplify the formula.

$$\begin{aligned}
\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t) &= -\frac{2}{n\pi}\int_0^t A'(s)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C \\
&= -\frac{2}{n\pi}\left[A(s)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)\Big|_0^t - \int_0^t A(s)\frac{d}{ds}\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds\right] + C \\
&= -\frac{2}{n\pi}\left[A(t)\exp\left(\frac{kn^2\pi^2}{L^2}t\right) - A(0) - \int_0^t A(s)\left(\frac{kn^2\pi^2}{L^2}\right)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds\right] + C \\
&= -\frac{2}{n\pi}\left[A(t)\exp\left(\frac{kn^2\pi^2}{L^2}t\right) - A(0) - \frac{kn^2\pi^2}{L^2}\int_0^t A(s)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds\right] + C
\end{aligned}$$

Set $t = 0$ and use the formula found for $B_n(0)$ earlier.

$$B_n(0) = C = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx - \frac{2}{n\pi}A(0)$$

Substitute this formula for C into the previous one and simplify it.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t) = -\frac{2}{n\pi}\left[A(t)\exp\left(\frac{kn^2\pi^2}{L^2}t\right) - \frac{kn^2\pi^2}{L^2}\int_0^t A(s)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds\right] + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

Solve for $B_n(t)$.

$$B_n(t) = -\frac{2}{n\pi}A(t) + \left[\frac{2kn\pi}{L^2}\int_0^t A(s)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)$$

The solution for w is then

$$\begin{aligned}
w(x,t) &= \sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty}\left\{-\frac{2}{n\pi}A(t) + \left[\frac{2kn\pi}{L^2}\int_0^t A(s)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\right\}\sin\frac{n\pi x}{L}.
\end{aligned}$$

Therefore, since $u(x,t) = w(x,t) + A(t)(1 - x/L)$,

$$\begin{aligned}
u(x,t) &= A(t)\left(1 - \frac{x}{L}\right) \\
&\quad + \frac{2}{L}\sum_{n=1}^{\infty}\left\{\left[\frac{kn\pi}{L}\int_0^t A(s)\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + \int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right) - \frac{L}{n\pi}A(t)\right\}\sin\frac{n\pi x}{L}.
\end{aligned}$$